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LETTER TO THE EDITOR

Multiparameter dependent solutions of the Yang–Baxter equation

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Abstract. A general method is developed for constructing representations of the Temperley–Lieb algebra, which in turn yield solutions of the Yang–Baxter equation. Several classes of multiparameter dependent R-matrices are obtained using this method, and the connection of the method with quantum groups is discussed.

Yang–Baxter equations (YBEs) play important roles in many branches of theoretical physics and pure mathematics. It has long been known that they underlie the basic structures of both classical and quantum mechanical integrable systems. Recent research has also revealed that they naturally appear in the theory of Lie bi-algebras and Hopf algebras as well as knot theory; and this has already led to significant advances in all these areas.

For physical applications, one is mainly interested in concrete solutions of YBEs; thus an important problem in the study of YBEs is to explicitly construct and systematically characterize, i.e. to classify, their solutions. This classification has been achieved for classical YBE solutions associated with simple Lie algebras [1] and Lie superalgebras [2]. In the quantum mechanical case, a complete classification of the YBE solutions appears to be very difficult, but a large number of classes of solutions are known, some of which have been thoroughly studied in statistical mechanics [3], and the recently developed techniques of quantum groups [4] and supergroups ([5], see Zachos (1990) for a review and a more complete list of references) offer a systematical way to construct trigonometric solutions.

In this letter we will construct several classes of multiparameter dependent solutions of YBE by studying representation of the Temperley–Lieb algebra (TLA) [6]. TLA is a unital algebra generated by U_i, $i = 1, 2, \ldots, N-1$, subject to the following constraint:

$$\begin{aligned} &U_i^2 = \sqrt{Q}\, U_i && Q \in \mathbb{C} \\ &U_i U_j = U_j U_i && |i-j| > 1 \\ &U_i U_{i\pm1} U_i = U_i. && \end{aligned} \tag{1}$$

Its connection with YBE is well known [6, 7]. Assume that we have a representation τ of TLA such that

$$\tau(U_i) = \tau_i = \underbrace{I \otimes \ldots \otimes I}_{i-1} \otimes T \otimes \underbrace{I \otimes \ldots \otimes I}_{N-i-1} \tag{2}$$

where $I \in \mathrm{End}(\mathbb{C}^m)$ is the identity matrix and $T \in \mathrm{End}(\mathbb{C}^m \otimes \mathbb{C}^m)$ will be referred to as a T-matrix. Then the matrix $\check{R}(u)$ defined by

$$\check{R}(u) = \frac{\sinh(\eta - u)}{\sinh \eta} + \frac{\sinh u}{\sinh \eta} T \tag{3}$$

satisfies the YBE, where η is a complex parameter related to Q through

$$2 \cosh \eta = \sqrt{Q}.$$

Therefore each representation of TLA of the form (2) yields a solution of YBE, and such TLA representations can be constructed using the representation theory of Lie algebra, quantum groups [8] and their $\mathbb{Z}_2$-graded counterparts [9].

We will develop a more general method for constructing T-matrices, and, therefore, solutions of YBE. Several classes of T-matrices are obtained using this method, which depend on multiples of free parameters, and include those T-matrices arising from Lie algebra and quantum groups as special cases.

The structure of the letter is as follows. First we develop the general method and then apply it to construct R-matrices; we then discuss the connection of one class of the R-matrices obtained with quantum groups and also use them to construct link polynomials. We conclude the letter by briefly summarizing the results and also mentioning several issues worth pursuing further.

Consider a matrix $T \in \mathrm{End}(\mathbb{C}^m \otimes \mathbb{C}^m)$. In order for it to generate a TLA representation through equation (2), it is necessary and sufficient that T satisfies the following equations

$$\begin{aligned} &T^2 = \sqrt{Q}\, T \\ &(I \otimes T)(T \otimes I)(I \otimes T) = I \otimes T \\ &(T \otimes I)(I \otimes T)(T \otimes I) = T \otimes I. \end{aligned} \tag{4}$$

In this letter we will study the T-matrices of the special form

$$T = \sum_{\mu,\nu,\sigma,\rho=1}^{m} g_{\mu\nu} h_{\rho\sigma} E_{\mu\sigma} \otimes E_{\nu\rho} \tag{5}$$

where $E_{\mu\nu} \in \mathrm{End}(\mathbb{C}^m)$ are the standard matrices such that the element of $E_{\mu\nu}$ at the position (μ, ν) is 1 while all the others vanish, and $g_{\mu\nu}$, $h_{\mu\nu}$ are the elements of the matrices g, $h \in \mathrm{End}(\mathbb{C}^m)$. Using the fact that

$$E_{\mu\nu} E_{\sigma\rho} = \delta_{\nu\sigma} E_{\mu\rho}$$

we can easily show that for the matrix T defined by (5),

$$\begin{aligned} &T^2 = \mathrm{tr}(gh)\, T \\ &(T \otimes I)(I \otimes T)(T \otimes I) = T \otimes g^{\mathrm{t}} h g h^{\mathrm{t}} \\ &(I \otimes T)(T \otimes I)(I \otimes T) = h^{\mathrm{t}} g h g^{\mathrm{t}} \otimes T \end{aligned} \tag{4$'$}$$

where g^{t} and h^{t} are respectively the transposes of g and h. Comparing (4′) and (4) we immediately see that the necessary and sufficient condition for (5) to qualify as a T-matrix is

$$\mathrm{tr}(gh) = \sqrt{Q} \neq 0 \qquad (gh)^{\mathrm{t}} = (hg)^{-1}. \tag{6}$$

We say that two matrices g and h form a (g, h)-pair if they satisfy equation (6). Thus the task facing us now is to construct as many as possible (g, h)-pairs. But before getting to that we examine the gauge symmetries of equations (4), (4′) and (6).

It is obvious that equation (4) retains its form under the transformation

$$T' = X \otimes XTX^{-1} \otimes X^{-1} \qquad X \in \mathrm{GL}(m, \mathbb{C}). \tag{7}$$

Thus T' will generate a TLA representation if T does. We can regard T and T' as equivalent, as the R-matrices associated with them through equation (3) yield the same statistical mechanics system. Now for a T-matrix of the form (5), we have

$$T' = X \otimes XTX^{-1} \otimes X^{-1} = \sum (XgX^{\mathrm{t}})_{\mu\nu} (X^{-\mathrm{t}} h X^{-1})_{\rho\sigma} E_{\mu\sigma} \otimes E_{\nu\rho} \tag{8}$$

where $X^{-\mathrm{t}}$ denotes $(X^{-1})^{\mathrm{t}} = (X^{\mathrm{t}})^{-1}$. Therefore the pairs (g, h) and (g', h') give rise to equivalent T-matrices if they are related to each other through

$$(g', h') = (XgX^{\mathrm{t}}, X^{-\mathrm{t}} h X^{-1}) \tag{9}$$

for some $X \in \mathrm{GL}(m, \mathbb{C})$.

An immediate consequence of the above discussion is that we can always assume that $g \in \mathrm{SL}(m, \mathbb{C})$. Then $\det h = \pm 1$, due to the second equation of (6). Now we consider concrete solutions of equation (6).

(1) One obvious class of solutions of (6) is obtained by setting $h = g^{-1}$, $g \in \mathrm{SL}(m, \mathbb{C})$. Now

$$\sqrt{Q} = \mathrm{tr}(gg^{-1}) = m$$

and

$$T = \sum g_{\mu\nu} (g^{-1})_{\rho\sigma} E_{\mu\sigma} \otimes E_{\nu\rho} \qquad g \in \mathrm{SL}(m, \mathbb{C}) \tag{10}$$

depends on $m^2 - 1$ complex parameters. The corresponding R-matrix reads

$$\check{R}(u) = \frac{\sinh(\eta - u)}{\sinh \eta} + \frac{\sinh u}{\sinh \eta} \sum g_{\mu\nu} (g^{-1})_{\rho\sigma} E_{\mu\sigma} \otimes E_{\nu\rho} \qquad 2\cosh \eta = m. \tag{11}$$

As is well known, given a self-dual finite-dimensional irreducible g-module V, where g is a simple Lie algebra, $V \otimes V$ contains a trivial g-module V_0, and the projection operator $P_0 \colon V \otimes V \to V_0$ leads to a T-matrix

$$T = \dim V \cdot P_0.$$

Such T-matrices belong to the subclass of those given in (10) with $g \in \mathrm{SO}(m, \mathbb{R})$.

(2) The next class of solutions of (6) is obtained by setting $h = g^{-\mathrm{t}}$. Now

$$\sqrt{Q} = \mathrm{tr}(gg^{-\mathrm{t}})$$

and we assume that $\sqrt{Q} \neq 0$. The corresponding R-matrix reads

$$\check{R}(u) = \frac{\sinh(\eta - u)}{\sinh \eta} + \frac{\sinh u}{\sinh \eta} \sum g_{\mu\nu} (g^{-\mathrm{t}})_{\rho\sigma} E_{\mu\sigma} \otimes E_{\nu\rho} \qquad 2\cosh \eta = \mathrm{tr}(gg^{-\mathrm{t}}) \tag{12}$$

which also depends on $m^2 - 1$ complex parameters as $g \in \mathrm{SL}(m, \mathbb{C})$. Note that the R-matrices (11) and (12) are not equivalent in general, as it is not always possible to transform (11) to (12) using the gauge symmetry (7). But when g is symmetric, they are obviously the same.

(3) Another class of (g, h)-pairs is obtained by requiring

$$\begin{aligned} &gh = hg \qquad g, h \in \mathrm{SO}(m, \mathbb{C}) \\ &\sqrt{Q} = \mathrm{tr}(gh) \neq 0. \end{aligned} \tag{13}$$

It is easy to see that equation (13) is indeed sufficient to ensure (6). Multiparameter dependent solutions of (13) can also be easily written down. When $m = 2$, $\mathrm{SO}(2, \mathbb{C})$ is Abelian, thus every pair of elements of $\mathrm{SO}(2, \mathbb{C})$ solves (13) as long as the trace of their product does not vanish. Explicitly,

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad h(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$\check{R}(u) = \frac{\sin(\theta+\varphi+iu)}{\sin(\theta+\varphi)} - \frac{\sin(iu)}{\sin(\theta+\varphi)} \sum g_{\mu\nu}(\theta) h_{\rho\sigma}(\varphi) E_{\mu\sigma} \otimes E_{\nu\rho} \qquad \theta, \varphi \in \mathbb{C}$$

thus we obtain an $\check{R}(u)$ which depends on two complex parameters θ and φ.

In higher dimensions, we consider as an example the $\mathrm{SO}(m, \mathbb{C})$ elements g and h which are of the following form

$$g = \begin{pmatrix} e & 0 \\ 0 & I \end{pmatrix} \qquad h = \begin{pmatrix} e^l & 0 \\ 0 & f \end{pmatrix}$$

where $e \in \mathrm{SO}(k, \mathbb{C})$, $1 < k < m$, $l \in \mathbb{Z}$, $f \in \mathrm{SO}(m-k, \mathbb{C})$. Again they trivially satisfy (13) provided that $\sqrt{Q} = \mathrm{tr}(e^{l+1}) + \mathrm{tr}(f) \neq 0$, and the corresponding R-matrix depends on $\frac{1}{2}k(k-1) + \frac{1}{2}(m-k)(m-k-1) + 1$ complex parameters.

Now we examine when will the R-matrices associated with the (g, h)-pairs defined by (13) be inequivalent. Define

$$H_g = \{h \in \mathrm{SO}(m, \mathbb{C}) | gh = hg\} \tag{14}$$

which obviously forms a group. Given any two elements h, $h' \in H_g$, if there exists an $X \in H_g$ such that

$$h = Xh'X^{-1}$$

which we denote by $h \sim h'$, then they give rise to equivalent R-matrices. Note that $\sim$ defines an equivalent relation; thus it makes sense to talk about the coset

$$C_g = H_g / \sim. \tag{15}$$

A representative of each element in C_g together with g constitutes a (g, h)-pair, and different elements of C_g lead to inequivalent T-matrices.

(4) Finally we note that given any $g \in \mathrm{SL}(m, \mathbb{C})$ and a symmetric matrix $S \in \mathrm{GL}(m, \mathbb{C})$, if

$$g = SgS \tag{16}$$

then the matrix defined by

$$h = S^{-1} g^{-1} \tag{17}$$

satisfies equation (16) with g provided that

$$\sqrt{Q} = \mathrm{tr}(S^{-1}) \neq 0.$$

When it is possible to express S as

$$S = e^D$$

the condition (16) says exactly that D and g anticommute, i.e.

$$\{D, g\} = Dg + gD = 0. \tag{18}$$

As $\det S = \pm 1$, $\mathrm{tr}\, D = ik\pi$, $k \in \mathbb{Z}$. It is worth pointing out that when $S = 1$, (16) is trivially satisfied, and we recover the first case studied.

Let us construct some concrete solutions of (16). In two dimensions, if

$$S = \begin{pmatrix} e^{\eta} & 0 \\ 0 & e^{-\eta} \end{pmatrix} \qquad 0 \neq \eta \in \mathbb{C} \tag{19a}$$

the most general solution of (16) is

$$g = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix} \qquad 0 \neq p \in \mathbb{C} \tag{19b}$$

and this in turn leads to

$$h = \begin{pmatrix} 0 & e^{-\eta}p^{-1} \\ e^{\eta}p & 0 \end{pmatrix} = \begin{pmatrix} 0 & q^{-1} \\ q & 0 \end{pmatrix} \tag{19c}$$

$$\sqrt{Q} = \mathrm{tr}(gh) = \frac{q}{p} + \frac{p}{q} = 2\cosh \eta. \tag{19d}$$

The corresponding R-matrix reads

$$\check{R}(u) = \frac{1}{\sinh \eta} \begin{pmatrix} \sinh(\eta - u) & 0 & 0 & 0 \\ 0 & e^{u} \sinh u & \sinh u/pq & 0 \\ 0 & pq \sinh u & e^{-u} \sinh u & 0 \\ 0 & 0 & 0 & \sinh(\eta - u) \end{pmatrix}. \tag{19e}$$

Note that when $pq = 1$, (19e) agrees with the modified six-vertex model R-matrix of [10].

Next we generalize the above example to higher dimensions. Let

$$S = e^D \qquad D = \begin{pmatrix} \mathbf{0} & & & & \\ & \eta_1\sigma_3 & & & \mathbf{0} \\ & & \eta_2\sigma_3 & & \\ & \mathbf{0} & & \ddots & \\ & & & & \eta_k\sigma_3 \end{pmatrix} \qquad 2k < m \tag{20a}$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad 0 \neq \eta_i \in \mathbb{C}.$$

A solution of (18) is

$$g=\begin{pmatrix} e & & & & & & & \\ & 0 & p_1^{-1} & & & & \mathbf{0} & \\ & p_1 & 0 & & & & & \\ & & 0 & p_2^{-1} & & & & \\ & & p_2 & 0 & & & & \\ & \mathbf{0} & & & \ddots & 0 & p_k^{-1} \\ & & & & & p_k & 0 \end{pmatrix} \qquad 0\neq P_i\in\mathbb{C} \tag{20b}$$

with e an arbitrary element of $\mathrm{SL}(m-2k,\mathbb{C})$. The corresponding h reads

$$h=e^{-D}g^{-1}=\begin{pmatrix} e^{-1} & & & & & & & \\ & 0 & q_1^{-1} & & & & \mathbf{0} & \\ & q_1 & 0 & & & & & \\ & & 0 & q_2^{-1} & & & & \\ & & q_2 & 0 & & & & \\ & & & & \ddots & & & \\ \mathbf{0} & & & & & 0 & q_k^{-1} \\ & & & & & q_k & 0 \end{pmatrix} \tag{20c}$$

where $q_i=e^{\eta_i}p_i$, and

$$\sqrt{Q}=\mathrm{tr}(gh)=2\sum_{i=1}^{k}\cosh\eta_i+m-2k. \tag{20d}$$

The (g, h)-pair given by equations $(20a$–$d)$ is almost the simplest we can think of; however, it covers all the T-matrices arising from self-dual multiplicity-free representations of quantum groups as special cases. We will now discuss the connection of the last class of (g, h)-pairs with quantum groups in more detail.

We study the relationship of the T-matrices arising from the fourth class of (g, h)-pairs discussed above with those from self-dual representations of quantum groups, then use them to construct link polynomials.

Recall that a quantum group $\mathrm{U}_q(g)$ [4] is a Hopf algebra, generated by $\{e_i, f_i, h_i \mid i=1,2,\ldots,r\}$, $r=$ rank of the simple Lie algebra g, subject to certain constraints. The co-product Δ: $\mathrm{U}_q(g)\to\mathrm{U}_q(g)\otimes\mathrm{U}_q(g)$ is defined by

$$\Delta(e_i)=e_i\otimes q^{h_i/2}+q^{-h_i/2}\otimes e_i$$
$$\Delta(f_i)=f_i\otimes q^{h_i/2}+q^{-h_i/2}\otimes f_i$$
$$\Delta(h_i)=h_i\otimes 1+1\otimes h_i.$$

Let π be an irreducible representation of $\mathrm{U}_q(g)$ furnished by a module V which is finite dimensional and self-dual. Then the tensor product $V\otimes V$ contains a trivial $\mathrm{U}_q(g)$-module V_0. Let P_0: $V\otimes V\to V_0$ be the projection operator mapping $V\otimes V$ onto V_0, and define [8]

$$T=D_q(V)P_0 \tag{21}$$

with $D_q(V)$ the q-dimension of V. Then T satisfies equation (4) with

$$\sqrt{Q}=D_q(V)$$

and thus qualifies as a T-matrix.

Since P_0 projects $V\otimes V$ onto a singlet, it can be written in the form

$$P_0=\sum_{\mu,\nu,\sigma,\rho=1}^{\dim V} e_{\mu\nu}f_{\rho\sigma}E_{\mu\sigma}\otimes E_{\nu\rho}/D_q(V). \tag{22}$$

It is easy to show, using the results of [11], that

$$\pi(q^{2h_\rho})/D_q(V)=\mathrm{tr}_2\{[\pi(q^{2h_\rho})\otimes\pi(q^{2h_\rho})]P_0\} \tag{23}$$

where tr_2 represents the partial trace taken over the second space in the tensor product, and h_ρ is an element of the Cartan subalgebra H of $\mathrm{U}_q(g)$ such that $\lambda(h_\rho)=(\rho,\lambda)$, $\forall\lambda\in H^*$, with $\rho\in H^*$, being the half-sum of the positive roots of g. It immediately follows (23) that

$$e\pi(q^{2h_\rho})f=1 \qquad ef=\pi(q^{2h_\rho})$$

i.e.

$$f=\pi(q^{-2h_\rho})e^{-1} \qquad e=\pi(q^{2h_\rho})e\pi(q^{2h_\rho}). \tag{24}$$

In a weight basis, $\pi(q^{2h_\rho})$ can always be written as

$$\pi(q^{-2h_\rho})=q^{2\pi(h_\rho)} \qquad \pi(h_\rho)=\begin{pmatrix} \mathbf{0} & & & & \\ & (\rho,\lambda_1)\sigma_3 & & & \mathbf{0} \\ & & (\rho,\lambda_2)\sigma_3 & & \\ & \mathbf{0} & & \ddots & \\ & & & & (\rho,\lambda_l)\sigma_3 \end{pmatrix} \tag{25}$$

where the λ_is are the positive weights of V, and not necessarily distinct when V is not multiplicity free. Therefore, all T-matrices of the form (21) are special cases of those arising from the fourth class of (g,h)-pairs discussed earlier.

It is worth emphasizing that the matrix e, uniquely determined by the quantum group structure, is a very special case of the solutions of (24) for a given $\pi(q^{2h_\rho})$. It involves only one free parameter q, while more general solutions of (24) can obviously contain multiples of free parameters, as demonstrated by the example given by equations ($20a$–d).

Now we turn to the construction of link polynomials using the following T- and R-matrices:

$$\begin{aligned} T&=\sum g_{\mu\nu}(S^{-1}g^{-1})_{\rho\sigma}E_{\mu\sigma}\otimes E_{\nu\rho}\\ \check{R}(u)&=\frac{\sinh(\eta-u)}{\sinh\eta}+\frac{\sinh u}{\sinh\eta}T \end{aligned} \tag{26}$$

where

$$g=SgS \qquad S=S^{\mathrm{t}} \qquad 2\cosh\eta=\mathrm{tr}(S^{-1}).$$

Define

$$\sigma=\lim_{u\to+\infty}\frac{-\sinh\eta}{\sinh u}\check{R}(u)$$

i.e.

$$\sigma=e^{-\eta}-T. \tag{27}$$

Then as a consequence of the Yang–Baxter equation,

$$\sigma_i = \underbrace{I\otimes\ldots\otimes I}_{i-1}\otimes\sigma\otimes\underbrace{I\otimes\ldots\otimes I}_{n-i-1} \qquad i=1,2,\ldots,n-1 \tag{28}$$

satisfy the defining relations of the Braid group B_n, thus afford a representation σ of B_n. It is well known (for a review, see [12]) that if we can define a Markov trace $\phi:\sigma(B_N)\to\mathbb{C}$, a link polynomial L can be obtained by letting

$$L(\theta)=(z\tilde{z})^{-(n-1)/2}\left(\frac{\tilde{z}}{z}\right)^{e(\theta)/2}\phi(\theta) \qquad \theta\in\sigma(B_n) \tag{29}$$

where $e(\theta)$ is the exponent sum of the word θ, and z, $\tilde{z}$ are defined by

$$\phi(\sigma_i)=z \qquad \phi(\sigma_i^{-1})=\tilde{z} \qquad \forall i. \tag{30}$$

Therefore our problem is to find a Markov trace ϕ on $\sigma(B_n)$. To do that, we follow the general method developed by Turaev in [13]. Note that

$$(S\otimes S)T=T(S\otimes S) \tag{31}$$

$$\begin{aligned}\mathrm{tr}_2[(S\otimes S)T] &= S\sum g_{\mu\nu}(S^{-1}g^{-1})_{\rho\sigma}E_{\mu\sigma}\,\mathrm{tr}(SE_{\nu\rho})\\ &= S\sum g_{\mu\nu}S_{\nu\rho}(S^{-1}g^{-1})_{\rho\sigma}E_{\mu\sigma}\\ &= S.\end{aligned} \tag{32}$$

Define $\phi:\sigma(B_n)\to\mathbb{C}$ by

$$\phi(\theta)=\mathrm{Tr}\Big\{(\underbrace{S\otimes\ldots\otimes S}_{n})\theta\Big\}\Big/\mathrm{Tr}(\underbrace{S\otimes\ldots\otimes S}_{n}) \qquad \theta\in\sigma(B_n) \tag{33}$$

where Tr denotes the trace taken over the n-fold tensor product space. Equations (31) and (33) guarantee that ϕ, satisfying the Markov properties, qualifies as a Markov trace, with

$$z=\phi(\sigma_i)=e^{-2\eta}(2\cosh\eta)^{-1} \qquad \tilde{z}=\phi(\sigma_i^{-1})=e^{2\eta}(2\cosh\eta)^{-1}. \tag{34}$$

In deriving (34), the fact that

$$\mathrm{tr}(S)=\mathrm{tr}(gS^{-1}g^{-1})=\mathrm{tr}(S^{-1})=2\cosh\eta$$

has been used. Applying equations (33) and (34) to (29) we obtain the following link polynomial

$$L(\theta)=(2\cosh\eta)^{n-1}e^{2\eta e(\theta)}\phi(\theta) \qquad \theta\in\sigma(B_{n-1}). \tag{35}$$

Since σ satisfies the second-order polynomial identity

$$\sigma-\sigma^{-1}=-2\sinh\eta$$

we see that $L(\theta)$ obeys the following Skein relation

$$\begin{aligned}&e^{-2\eta}L(\theta_1\sigma_i\theta_2)-e^{2\eta}L(\theta_1\sigma_i^{-1}\theta_2)=-2\sinh\eta\,L(\theta_1\theta_2)\\ &\forall\theta_1,\theta_2\in\sigma(B_n) \qquad i=1,2,\ldots,n-1.\end{aligned} \tag{36a}$$

Also, for the unknotted knot, $\theta=\sigma_1\in\sigma(B_2)$, we have

$$L=1 \qquad \text{for the unknotted knot.} \tag{36b}$$

It follows a theorem of Kauffman [14] that equation (36) uniquely determines a link polynomial, which agrees with the Jones polynomial.

We have presented a method for constructing representations of the Temperley–Lieb algebra and, thus, solutions of the Yang–Baxter equation. Several classes of TLA representations and R-matrices are obtained, which depend on families of free parameters. As we demonstrated, one class of them covers all the TLA representations arising from self-dual representations of quantum groups. Using the R-matrices associated with this class of TLA representations, we have also obtained the Jones polynomial.

A question we did not touch upon in this letter is the physical meaning of the free parameters in the R-matrices obtained. This can only be answered by explicitly working out the corresponding statistical mechanics models, and we plan to do this in the future. Another problem is the role played by the free parameters in the deformation theory of Lie groups and Lie algebras. Using, for example, the R-matrix associated with the (g, h)-pair given by equation (19), one can build a quantum group following the procedure developed by Faddeev *et al* [15]. It will be very interesting to see how the resultant quantum group is related to the standard $U_q(\mathrm{sl}(2))$. Finally, it might be possible to classify the solutions of (6) more thoroughly. It is a remarkably simple equation; we should be able to say something more general about the structure of its solutions.

References

[1] Belavin A A and Drinfeld V G 1983 *Funct. Anal. Appl.* **16** 159

[2] Leites D A and Serganova V V 1984 *Theor. Math. Phys.* **58** 16

[3] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

[4] Drinfeld V G 1987 *Proc. ICM, Berkeley* p 798
Jimbo M 1985 *Lett. Math. Phys.* **10** 63

[5] Kulish P P and Reshetikhin N Yu 1989 *Lett. Math. Phys.* **18** 143
Chaichian M and Kulish P P 1990 *Phys. Lett.* **234B** 72
Bracken A J, Gould M D and Zhang R B 1990 *Mod. Phys. Lett.* A **5** 72
Zhang R B, Bracken A J and Gould M D 1991 Solution of the graded Yang–Baxter equation associated with the vector representation of $U_q(\mathrm{osp}(M/2n))$ *Phys. Lett.* B in press
Deguchi T, Fujii A and Ito K 1990 *Phys. Lett.* **238B** 242
Saleur H 1990 *Nucl. Phys.* B **336** 363
Curtright J and Ghandour G 1989 *Miami Preprint TM/8/89*
Zachos C 1990 Paradigms of quantum groups *Argonne Preprint ANL-HEP-PR-90-61*

[6] Temperley H V and Lieb E H 1971 *Proc. R. Soc.* A **322** 251

[7] Baxter R J 1982 *J. Stat. Phys.* **28** 1
Owczarek A L and Baxter R J 1987 *J. Stat. Phys.* **49** 1093
Batchelor M T and Barber M N 1990 *J. Phys. A: Math. Gen.* **23** L15

[8] Batchelor M T and Kuniba A 1990 Temperley-Lieb lattice models arising from quantum groups *Australian National University Preprint*

[9] Zhang R B 1991 Graded representations of the Temperley-Lieb algebra, quantum supergroups and the Jones polynomial *University of Queensland Preprint UQMATH-91-2*

[10] Babelon O, de Vega H J and Viallet C 1981 *Nucl. Phys.* B **190** [F53] 542

[11] Zhang R B, Gould M D and Bracken A J 1991 Quantum group invariants and link polynomials *Commun. Math. Phys.* in press

[12] Wadati M, Deguchi T and Akutsu Y 1989 *Phys. Rep.* **180** 248

[13] Turaev V G 1988 *Invent. Math.* **92** 527

[14] Kauffman L 1987 *Topology* **26** 395

[15] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1988 *Algebra and Analysis* **1** 129